## ИНФОРМАЦИОННЫЕ СИСТЕМЫ И ТЕХНОЛОГИИ
## INFORMATION SYSTEM AND TECHNOLOGIES

**UDC 004.855.5** **DOI: 10.18413/2518-1092-2024-9-3-0-1**

**Konstantinov I.S.**[1]
**Taha A.T.T.**[2]

# MATHEMATICAL ANALYSIS OF SIR MODEL WITH INCUBATION PERIOD

[1)] Belgorod State Technological University named after V.G. Shukhov,
46 Kostyukova St., Belgorod, 308012, Russia
[2)] Belgorod State National Research University,
85 Pobedy St., Belgorod, 308015, Russia

*e-mail: konstantinov@bsu.edu.ru*

**Abstract**
The SIP model is a fundamental tool for simulating epidemic dynamics, but it has limitations in accurately representing real-world scenarios. This paper presents a comprehensive review and mathematical analysis of the SIR model when incorporating the incubation period of infectious diseases. We discuss the significance of the incubation period in disease transmission dynamics and explore the modifications made to the SIR model to integrate this parameter. By analyzing the mathematical equations governing the modified SIR model, we demonstrate its enhanced accuracy in predicting disease spread patterns and its implications for public health interventions. Our findings evaluated the importance of incorporating the incubation period into epidemic models.
**Keywords:** SIR model; incubation period; epidemic dynamics; disease transmission; mathematical modeling
**For citation:** Konstantinov I.S., Taha A.T.T. Mathematical analysis of SIR model with incubation period // Research result. Information technologies. – T.9, №3, 2024. – P. 3-9. DOI: 10.18413/2518-1092-2024-9-3-0-1

**Константинов И.С.**[1]
**Таха А.Т.Т.**[2]

# МАТЕМАТИЧЕСКИЙ АНАЛИЗ МОДЕЛИ SIR С УЧЕТОМ ПЕРИОДА ИНКУБАЦИИ

[1)] Белгородский государственный технологический университет им. В.Г. Шухова,
ул. Костюкова, 46, Белгород, 308012, Россия
[2)] Белгородский государственный национальный исследовательский университет,
ул. Победы, 85, г. Белгород, 308015, Россия

*e-mail: konstantinov@bsu.edu.ru*

**Аннотация**
Модель SIR является фундаментальным инструментом для моделирования эпидемической динамики, но у нее есть ограничения в точном представлении реальных сценариев. В данной статье представлено всестороннее обзорное и математическое аналитическое исследование модели SIR при включении инкубационного периода инфекционных заболеваний. Мы обсуждаем значимость инкубационного периода в динамике передачи заболевания и исследуем модификации, внесенные в модель SIR для интеграции этого параметра. Анализируя математические уравнения, определяющие измененную модель SIR, мы демонстрируем ее улучшенную точность в прогнозировании паттернов распространения болезни и ее влияние на меры общественного здравоохранения. Наши результаты подчеркивают важность включения инкубационного периода в эпидемические модели.
**Ключевые слова**: модель SIR; инкубационный период; эпидемическая динамика; передача заболевания; математическое моделирование

**Для цитирования:** Константинов И.С., Таха А.Т.Т. Математический анализ модели SIR с учетом периода инкубации // Научный результат. Информационные технологии. – Т.9, №3, 2024. – С. 3-9. DOI: 10.18413/2518-1092-2024-9-3-0-1

## *INTRODUCTION*

The original SIR model is a fundamental mathematical framework used to study the dynamics of infectious diseases within populations. It divides the population into three compartments based on their disease status: Susceptible (S), Infectious (I), and Removed (R). Here's a focused and informative explanation:

The Susceptible (S) compartment represents individuals who are susceptible to the disease and can become infected upon contact with infectious individuals. The Infectious (I) compartment consists of individuals who are currently infected and capable of transmitting the disease to susceptible individuals. The Removed (R) compartment includes individuals who have either recovered from the disease and gained immunity or have been removed from the population due to death or other reasons. The dynamics of the SIR model are governed by a set of ordinary differential equations (ODEs) that describe how the number of individuals in each compartment changes over time. These equations capture the flow of individuals between compartments based on rates of infection, recovery, and removal.

In Fig. 1 The incubation period refers to the duration between exposure to a pathogen and the onset of symptoms in an infected individual. It represents the time taken for the pathogen to multiply and cause detectable signs of illness. In contrast, the latent period specifically denotes the time between infection and the individual becoming infectious to others.

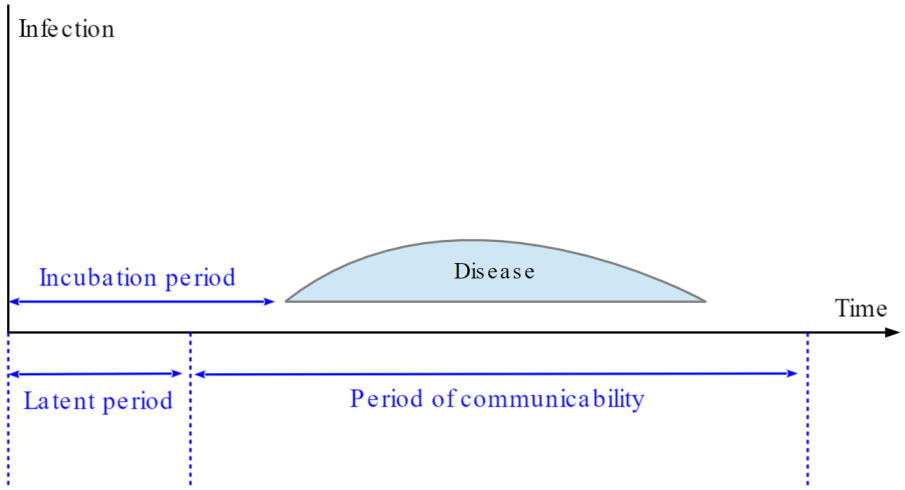


*Fig. 1.* Incubation Period Explained
*Рис. 1.* Инкубационный период – пояснение

Mathematically, the differential equations for the SIR model can be represented as follows:

$$\begin{cases} dt/dS = -\beta SI \\ dt/dI = \beta SI - \gamma I \\ dt/dR = \gamma I \end{cases} \quad (1)$$

Where:

$S, I,$ and $R$ represent the numbers of individuals in the Susceptible, Infectious, and Removed compartments, respectively.

$N$ is the total population size ($N = S + I + R$).

$\beta$ is the transmission rate, representing the probability of disease transmission from an infectious individual to a susceptible individual per unit time.

$\gamma$ is the recovery rate, representing the rate at which infectious individuals recover and move to the Removed compartment.

## *LITERATURE REVIEW*

The SIR model, commonly used for simulating epidemics, has several limitations. Firstly, accurately inferring model parameters based on early, noisy observations are problematic, limiting its practical identifiability [9]. Secondly, the SIR model assumes equal susceptibility and connectivity within age bands, leading to overestimation of infection rates [11]. Additionally, the model fails to consider the role of nosocomial transmission in overloading health systems, resulting in inaccurate predictions of critical care requirements [8]. Furthermore, the SIR model does not account for heterogeneity in contact rates, which affects the stability of the epidemic spread in real-world scenarios [2].

The incubation period of infectious diseases plays a crucial role in understanding their dynamics. Differences in the incubation period can determine the predictability and spread patterns of diseases [3]. Diseases with longer incubation periods, such as Ebola, tend to have less predictable spread and more long-distance sparking events[3]

Incorporating the incubation period into the SIR model enhances disease spread modeling accuracy by accounting for the delay between infection and detection [14]. Understanding the incubation period aids in designing effective quarantine and isolation durations, impacting disease control strategies. Moreover, considering the incubation period's influence on social distancing measures reveals insights into optimal control strategies, emphasizing timing and duration for effective disease containment [12].

Incorporating the incubation period into the SIR model is epidemiologically significant as it has implications for disease forecasting, intervention planning, and public health policy. The length of the incubation period is strongly correlated with disease severity, and accurately estimating the incubation time ranges can help establish appropriate quarantine durations, aiding in controlling future outbreaks[10].Validation studies have been conducted to assess the accuracy of the modified SIR model with an incubation period in simulating real-world disease outbreaks. These studies have demonstrated the ability of the modified SIR model to accurately predict the number of beds needed during the early stages of an epidemic, such as the COVID-19 outbreak in Wuhan[6,7].

The modified SIR model is compared with the classical SIR model in terms of disease dynamics and control strategies. The modified model incorporates network structure centrality measures and illness factors, while the classical model does not consider these factors. The modified model is found to perform better in terms of lowering the final death ratio in the community [1]. Sensitivity analyses have been conducted to assess the impact of varying the duration of the incubation period on model outcomes in several studies. Sun discusses the importance of good practice sensitivity analysis, including the consideration of robustness to choices in methods and assumptions, and the need for convergence of sensitivity metrics[16].

The incorporation of the incubation period into the SIR model has yielded valuable insights for specific infectious diseases such as COVID-19. Liu et al. quantified data irregularity using approximate entropy and found higher volatility in the U.S., Italy, and India compared to China 5] . Computational methods used to simulate the modified SIR model with an incubation period involve agent-based modeling (ABM) and numerical integration techniques such as Euler's method [17]. The SIR model for incorporating the incubation period has limitations and assumptions. One limitation is the assumption of homogeneous mixing, which assumes that individuals have an equal chance of coming into contact with each other. This assumption may not hold in real-world scenarios where individuals have different contact patterns. Another limitation is the assumption of constant transmission rates, which assumes that the rate at which individuals become infected remains constant over time[13].

Future research in infectious disease modeling should focus on refining the modified SIR model by incorporating additional factors and integrating real-time data sources. This will enhance the accuracy and predictive power of the model, allowing for more effective public health decision-making and outbreak response strategies. Incorporating the incubation period into the SIR model has important implications for public health. It can provide valuable insights into the timing and progression of outbreaks, enabling the development of targeted control measures [6,7]. Numerous lectures, hosted by the Oxford University

Department for Continuing Education, have explored the dynamics of the SIR model, which researchers have utilized in this paper [18].

## *MODEL DEVELOPMENT*

Now, we make an additional assumption to equation 1: the disease has an incubation period equal to $\tau$. This means that a susceptible individual who comes into contact with an infective and contracts the disease would not actually move to the infectious category for a certain period of time labeled $\tau$. We can think of this as someone catching the disease from an infectious person, but taking some time before displaying symptoms and therefore becoming infectious to others. What this means for our model is that variables I and S, which were originally functions of time t, will now become functions of (t-$\tau$). Our system would appear as follows in equation 2

$$\begin{cases} \frac{dS}{dt} = -rS(\mathrm{t}-\tau)I(\mathrm{t}-\tau) \\ \frac{dI}{dt} = rS(\mathrm{t}-\tau)I(\mathrm{t}-\tau) - aI(t) \\ \frac{dR}{dt} = aI(t) \end{cases} . \quad (2)$$

As noticed, $aI$ hasn't been changed to $aI(t-\tau)$ because the assumption of the incubation period applies only when transitioning from S to I. The movement from I to R, representing the recovery phase, does not incorporate the delay period. In the context of COVID-19, the incubation period is believed to be around five days, according to the WHO.

In Fig. 2, after simulation of equation 2, we noticed that the entire graph shifts in time as you increase the incubation period above zero, while the dynamics remain fundamentally the same.

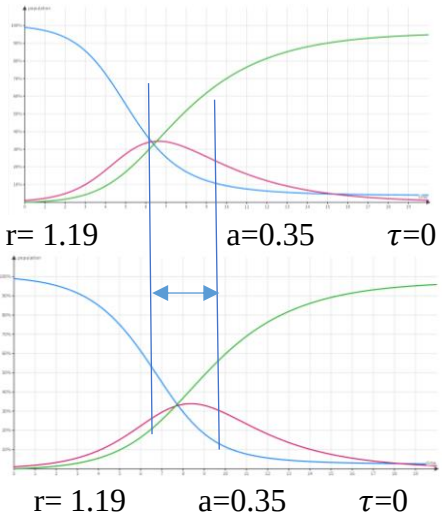


*Fig. 2.* Classical SIR Model vs Incubation Period Enabled SIR Model
*Рис. 2.* Классическая модель SIR и модель SIR с включенным инкубационным периодом

In the context of COVID-19, the incubation period is relatively short compared to the timescale of the disease. This holds true for any disease modeled using the SIR method. Therefore, we have a system of equations containing a small parameter, suggesting the use of Taylor expansion. In other words, we aim to simplify system equation3 2 into a more manageable form. Thus, we expand equation 1 around t, assuming that τ is a small value.

$$rS(t-\tau)\mathrm{I}(\mathrm{t}-\tau) \approx \mathrm{rS(t)I(t)} + \mathrm{r}\tau(\mathrm{SI})' + \mathrm{O}(\tau^2) \quad (3)$$

What would we do next is take our approximation in equation 3 and substitute it into equations system in 2.

$$\frac{dS}{dt} = -\mathrm{rS(t)I(t)} - r\tau(IS)' + O(\tau^2) \quad (4)$$

$$\frac{dI}{dt} = rIS - aI + r\tau(IS)' + O(\tau^2) \quad (5)$$

$$\frac{dR}{dt} = aI(t) \quad (6)$$

The $\tau$ term which was originally inside our unknown variables through Taylor expansion approximation for a small $\tau$ we've been write it Infront of something we can now workout. the next step involves pretty heavy algebra which is take $(IS)'$ term and simplify them into something more manageable. we would do this by using the fact that equations in 2 is originally has S' and I', they can be substituted for S' and I' and rearrange to get a simplified expression for $(IS)'$

$$(IS)' = \frac{IS(r(S-I)-a)}{1-r\tau(I-S)} \quad (7)$$

We can call $\frac{(r(S-I)-a)}{1-r\tau(I-S)}$ as a $f(r,\tau,a,I,S)$, so the equation 7 would be

$$(IS)' = IS\, f(r,\tau,a,I,S) \quad (8)$$

Now we will neglect $O(\tau^2)$ in equation 4 and 5 and substitute equation 8 in them

$$\frac{dS}{dt} = -rIS - r\tau IS\, f(r,\tau,a,I,S) \quad (9)$$

$$\frac{dI}{dt} = rIS - aI + r\tau IS\, f(r,\tau,a,I,S) \quad (10)$$

Now we would combine equation 9 and equation 10 by dividing $\frac{dI}{dt}$ by $\frac{dS}{dt}$ to have $\frac{dI}{dS}$

$$\frac{dI}{dS} = \frac{rIS-aI+\tau rISf}{-rIS(1+\tau f)} \quad (11)$$

As we mentioned $\tau$ is a small parameter, so we can ignore $\tau^2$, so we can make another expansion of the term $(1+\tau f)^{-1}$ , by polynomial expansion, geometric series or Taylor expansion, which says if one plus something small can be approximated to:

$$(1+\tau f)^{-1} \approx 1 - \tau f + O(\tau^2) \quad (12)$$

Now we can substitute equation 12 in equation 11, so:

$$\frac{dI}{dS} = \frac{-1}{rIS}(rIS - aI + \cancel{\tau rISf} - \cancel{\tau rISf} + a\tau If) + O(\tau^2) \quad (13)$$

So, we can rewrite equation 13 as the following

$$\frac{dI}{dS} = \frac{-(rIS-aI)}{rIS} + O(a\tau,\tau^2) \quad (14)$$

According to [19], the data where as shown in Fig. 3, so we need to keep hard working to build a mathematical model that govern the speed of pandemic, so we can be ready to the next round when disease hits again

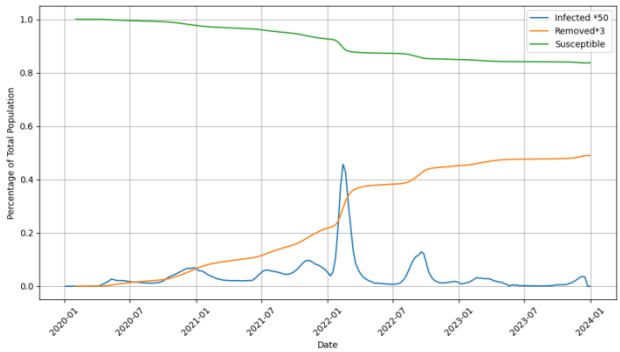


*Fig. 3.* Real-World Data SIR Model for COVID-19 Dynamics in Russia
*Рис. 3.* Модель SIR с реальными данными о динамике распространения COVID-19 в России

## *CONCLUSION*

The equation 14 is the exact to that in SIR model without incubation period except the term $O(a\tau,\tau^2)$, so we can conclude that our model remains unchanged as we can neglect the terms of $O(a\tau,\tau^2)$, and that explains why the shape is not changed when incubation time is changed and just is shifted in time. That is mean in practice, even if we include the incubation time in SIR model, the structure and the behavior of the system will actually be the same as the basic SIR model. For a disease like COVID19 where incubation

period is indeed is small, it is very reasonable to ignore the term $O(a\tau, \tau^2)$. This tells us when we trying to model a pandemic, and trying to understand the spread of the disease. we can simplify our model and we don't need to worry about the incubation period because the dynamics would be the same. what we can see the contact ratio is the key parameter to control the dynamic of the disease.

**References**

**Список литературы**

1. Chatterjee S., Zehmakan A.N. (2023). Effective Vaccination Strategies in Network-based SIR Model. http://arxiv.org/abs/2305.16458
2. Hansson J., Govaert A., Pates R., Tegling E., Soltesz K. (2022). Limitations of time-delayed case isolation in heterogeneous SIR models. https://doi.org/10.23919/ACC53348.2022.9867465
3. Kahn R., Peak C.M., Fernández-Gracia J., Hill A., Int H., Jambai A., Ganda L., Castro M.C., Buckee C.O., Edu C.H. (n.d.). Incubation periods impact the spatial predictability of outbreaks: analysis of cholera and Ebola outbreaks in Sierra Leone. https://doi.org/10.1101/19003525
4. Kahn R., Peak C.M., Fernández-Gracia J., Hill A., Jambai A., Ganda L., Castro M. C., Buckee C.O. (n.d.). World Health Organization, CH-1211 Geneva 27, Switzerland; d Disease Control and Prevention, Sierra Leone Ministry of Health and Sanitation. https://doi.org/10.1073/pnas.1913052117/-/DCSupplemental
5. Liu J., Tian B., Wu J. 2023. Temporal Analysis of Infectious Diseases: A Case Study on COVID-19. Acadlore Transactions on Applied Mathematics and Statistics, 1(1), 1–9. https://doi.org/10.56578/atams010101
6. Liu T., Huang J., He Z., Zhang Y., Yan N., Zhang C.J.P., Ming W.K. (2023a). A real-world data validation of the value of early-stage SIR modelling to public health. Scientific Reports, 13(1). https://doi.org/10.1038/s41598-023-36386-9
7. Liu T., Huang J., He Z., Zhang Y., Yan N., Zhang C..JP., Ming W.K. (2023b). A real-world data validation of the value of early-stage SIR modelling to public health. Scientific Reports, 13(1). https://doi.org/10.1038/s41598-023-36386-9
8. Mckeigue P.M., Wood S.N. (2022). Limitations of models for guiding policy in the COVID-19 pandemic. https://doi.org/10.1101/2022.06.30.22277091
9. Melikechi O., Young A. L., Tang T., Bowman T., Dunson D., Johndrow J. (2022). Limits of epidemic prediction using SIR models. Journal of Mathematical Biology, 85(4). https://doi.org/10.1007/s00285-022-01804-5
10. Owen L., Hoseana, J., & Yong B. (2023). Codimension-Two Bifurcations of an SIR-Type Model for COVID-19 and Their Epidemiological Implications. http://arxiv.org/abs/2307.08892
11. Salman A.M., Hafiz Mohd M., Ahmad N.A., Musa I., Ahmed I., Alqahtani Z. (2022). An Optimal Control of SIRS Model with Limited Medical Resources and Reinfection Problems. In Malaysian Journal of Fundamental and Applied Sciences (Vol. 18).
12. Sharbayta S.S., Buonomo B., d'Onofrio A., Abdi T. (2022). 'Period doubling' induced by optimal control in a behavioral SIR epidemic model. Chaos, Solitons & Fractals, 161, 112347.
13. Shatz I. (2023). Assumption-checking rather than (just) testing: The importance of visualization and effect size in statistical diagnostics. Behavior Research Methods. https://doi.org/10.3758/s13428-023-02072-x
14. Singh R.P. (2022). Bifurcation and Stability Analysis of Delayed SIR Model. Journal of Physics: Conference Series, 2267(1). https://doi.org/10.1088/1742-6596/2267/1/012011
15. SUN X. (2023). Sensitivity analysis: Advancing the effectiveness of global sensitivity analysis. Bulletin of the Australian Mathematical Society, 107(2), 351–352.
16. Taib N.A.A., Labadin J. (2022). Sensitivity Analysis of the Effective Reproduction Number in a Sarawak Rabies Epidemic Model. 2022 International Conference on Future Trends in Smart Communities, ICFTSC 2022, 194–198. https://doi.org/10.1109/ICFTSC57269.2022.10039940
17. Varghese A., Lacap Jr. E.M., Sajath I., Kumar M.K., Kolamban S. (2022). Controlling Epidemics with Mathematical and Machine Learning Models. IGI Global.
18. Oxford University Department for Continuing Education. (2021). "Pandemic Dynamics series: Dr. Tom Crawford". URL: https://www.conted.ox.ac.uk/profiles/tom-crawford
19. Our World in Data. (2023). "COVID-19 Data Explorer." Accessed December 29, 2023. URL: https://ourworldindata.org/covid-cases

**Konstantinov Igor Sergeyevich,** Doctor of Technical Sciences, Professor, Director of the Institute of Information Technologies and Control Systems
**Taha Asraa Tariq Taha,** Postgraduate of the Department of Mathematical and Software of Information Systems

**Константинов Игорь Сергеевич,** доктор технических наук, профессор, директор института информационных технологий и управляющих систем
**Таха Асраа Тарик Таха,** аспирант кафедры математического и программного обеспечения информационных